UDC 62-50

ON A SUFFICIENT CONDITION FOR NONLINEAR POSITION GAMES OF ENCOUNTER

PMM Vol. 40, № 1, 1976, pp. 168-171

A. G. PASHKOV
(Moscow)

(Received March 6, 1974)

We examine an encounter game problem for a nonlinear conflict-controlled system. We have shown that sufficient conditions for a successful completion of the nonlinear position encounter game problem being examined can be derived, under specified assumptions, from the regularity conditions of suitable differential games. As an example we analyze the solution of the geometric-coordinate encounter problem for two objects (material points) whose motions are described by quasi-linear differential equations; there is a "gap" in the control system of the first (pursuing) object. The paper is closely related to the investigations in [1 − 4].

1. We consider a conflict-controlled system described by the vector differential equation

$$x^{\cdot} = f(t, x, u, v), \quad u \in P, \quad v \in Q \tag{1.1}$$

Here x is the system's n-dimensional phase vector; u and v are r-dimensional control vectors of the first and second player, respectively; P and Q are closed bounded sets. The function $f(t, x, u, v)$ is assumed to be continuous in all arguments, satisfies Lipschitz conditions in x in each bounded domain of space $\{x\}$ and satisfies the conditions of extendibility of all solutions $x(t)$ of the contingency equation $x^{\cdot} \in F(t, x)$ for all $t \geqslant t_0$. Here $F(t, x) = \text{co}\, f(t, x, u, v)$ for $u \in P$ and $v \in Q$.

A closed set M — the first player's target — is given in space $\{x\}$. The initial position $\{t_0, x_0\}$ is fixed. An admissible strategy of the first player U is defined as the function which associates a closed set $U(t, x) \in P$ with every position $\{t, x\}$, where the sets $U(t, x)$ are upper semicontinuous relative to inclusion as the position $\{t, x\}$ changes. The motion $x[t] = x[t, t_*, x_*, U]$ is defined as any absolutely continuous function satisfying the conditions $x[t_*] = x_*$ and $x^{\cdot}[t] \subset F_U(t, x[t])$ for almost all $t \geqslant t_*$. Here $F_U(t, x) = \text{co}\,\{f(t, x, u, v) \mid u \in U(t, x), v \in Q\}$. By definition, the strategy U_ϑ guarantees the encounter of point $x[t]$ with target M from the position $\{t_0, x_0\}$ at an instant $\vartheta \geqslant t_0$, if $x[\vartheta] \in M$ for any motion $x[t] = x[t, t_0, x_0, U_\vartheta]$.

The synthesis problem for control u, operating on the feedback principle and guaranteeing the encounter of point $x[t]$ with set M under arbitrary actions of the second player $v \in Q$, based on any information he has, can be formulated in the form of the following encounter problem [1]. Find the instant $\vartheta \geqslant t_0$ and the strategy U_ϑ guaranteeing the encounter of point $x[t]$ with M from position $\{t_0, x_0\}$ at the instant ϑ.

We introduce an auxiliary system described by the vector differential equation

$$x^{\cdot} = \varphi(t, x) + u_1 - v_1, \quad u_1 \in P_1, \ v_1 \in Q_1 \tag{1.2}$$

Here x is the system's n-dimensional phase vector; u_1 and v_1 are the r-dimensional control vectors of the first and second players, respectively; P_1 and Q_1 are bounded, convex and closed sets. The function $\varphi(t, x)$ is assumed to be continuous in all arguments and to satisfy a Lipschitz condition in x.

We consider the following problem. Find the pair of controls $\{u_1^\circ(t), v_1^\circ(t)\}$ which yields a solution to the following program maximin problem for system (1.2)

$$\rho[x(\vartheta, t_*, x_*, u_1^\circ, v_1^\circ), M] = \max_{v_1(t)} \min_{u_1(t)} \rho[x(\vartheta, t_*, x_*, u_1, v_1), M] \qquad (1.3)$$

We denote $\varepsilon_0(t_*, x_*, \vartheta) = \rho[x(\vartheta, t_*, x_*, u_1^\circ, v_1^\circ), M]$. We assume that the regular case of the game of encounter with set M holds for system (1.2), i.e. for every position $\{t_*, x_*\}$, $t_* < \vartheta$, for which $0 < \varepsilon_0(t_*, x_*, \vartheta) < \delta$, $\delta > 0 = \text{const}$, problem (1.3) has a unique solution $\{u_1^\circ(t), v_1^\circ(t)\}$ and the point x_M of M, closest to the point $x(\vartheta, t_*, x_*, u_1^\circ, v_1^\circ)$, is unique as well. In the regular case of the encounter game for system (1.2), for a fixed value of ϑ the function $\varepsilon_0(t, x, \vartheta)$, continuous for $t \leqslant \vartheta$ and $0 < \varepsilon_0 < \delta$, is differentiable in the domain $t < \vartheta$ and $0 < \varepsilon_0 < \delta$ and satisfies the inequality [2]

$$\min_{u_1\in P_1} \max_{v_1\in Q_1} \frac{d\varepsilon_0}{dt} = \min_{u_1\in P_1} \max_{v_1\in Q_1} \left[\frac{\partial\varepsilon_0}{\partial t} + \left(\frac{\partial\varepsilon_0}{\partial x}\right)' (\varphi(t, x) + u_1 - v_1)\right] = \qquad (1.4)$$
$$\max_{v_1\in Q_1} \left[\frac{\partial\varepsilon_0}{\partial t} + \left(\frac{\partial\varepsilon_0}{\partial x}\right)' (\varphi(t, x) + u_{1e} - v_1)\right] \leqslant 0$$

We define the extremal strategy U_{e_1} for system (1.2) as follows: if $\varepsilon_0(t, x, \vartheta) = 0$ or $\varepsilon_0(t, x, \vartheta) \geqslant \delta$, or if $t \geqslant \vartheta$, then $U_{e_1}^{(\vartheta)} = P_1$; however, if $0 < \varepsilon_0(t, x, \vartheta) < \delta$ and $t < \vartheta$, then $U_{e_1}^{(\vartheta)}(t, x)$ is added together from all vectors $u_{e_1} \in P_1$ which satisfy the minimax condition

$$\max_{v_1\in Q_1} \left(\frac{\partial\varepsilon_0}{\partial x}\right)' (\varphi(t, x) + u_{1e} - v_1) = \min_{u_1\in P_1} \max_{v_1\in Q_1} \left(\frac{\partial\varepsilon_0}{\partial x}\right)' (\varphi(t, x) + u_1 - v_1) \qquad (1.5)$$

The following assertion can be proved analogously to the theorem in [1].

Let the encounter game for the motion of system (1.2) be regular. Then the extremal strategy $U_{e_1}^{(\vartheta)}$ is admissible. If from the initial position $\{t_0, x_0\}$ the set M is absorbed at an instant ϑ° in a program manner with respect to the minimax, i.e. if $\rho[x(\vartheta, t_0, x_0, u_1^\circ, v_1^\circ), M] = 0$, then the extremal strategy $U_{e_1}^{(\vartheta)}$ solves the problem of the encounter of the motion of (1.2) with set M.

We state the fundamental result as follows.

Theorem. Let the regular case of the game of encounter with a set hold for the auxiliary system (1.2) and let the relation

$$\min_{u_1\in P_1} \max_{v_1\in Q_1} s'(\varphi(t, x) + u_1 - v_1) \geqslant \min_{u\in P} \max_{v\in Q} s' f(t, x, u, v) \qquad (1.6)$$

where s is an arbitrary n-dimensional vector and the prime denotes transposition, be fulfilled. Then, if from the initial position $\{t_0, x_0\}$ the set M is absorbed by system (1.2) at an instant ϑ in a program manner with respect to the minimax, then the corresponding first player's strategy $U_e^{(\vartheta)}$ can be constructed, guaranteeing him the encounter of the motion of system (1.1) with M at instant ϑ.

Proof. With due regard to (1.4) and (1.5), from (1.6) follows

$$\min_{u\in P} \max_{v\in Q} \left[\frac{\partial\varepsilon_0}{\partial t} + \left(\frac{\partial\varepsilon_0}{\partial x}\right)' f(t, x, u, v)\right] \leqslant \qquad (1.7)$$

$$\leqslant \min_{u_1\in P_1}\max_{v_1\in Q_1}\left[\frac{\partial\varepsilon_0}{\partial t}+\left(\frac{\partial\varepsilon_0}{\partial x}\right)'(\varphi(t,x)+u_1-v_1)\right]\leqslant 0$$

Thus, in the domain $t\leqslant\vartheta$ we have constructed a continuous function $\varepsilon_0(t,x,\vartheta)$ satisfying the condition $\varepsilon_0(\vartheta,x,\vartheta)>0$ for $x(\vartheta)\in M$ and such that inequality (1. 7) is fulfilled for $t<\vartheta$ and $\varepsilon(t,x,\vartheta)>0$. We define the first player's strategy $U_e^{(\vartheta)}$ for the original system (1. 1) in the following manner: if $\varepsilon_0(t,x,\vartheta)=0$ or $\varepsilon_0(t,x,\vartheta)\geqslant\delta$, or if $t\geqslant\vartheta$, then $U_e^{(\vartheta)}=P$; however, if $0<\varepsilon_0(t,x,\vartheta)<\delta$ and $t<\vartheta$, then $U_e^{(\vartheta)}(t,x)$ is added together from the vectors $u_e\in P$ which satisfy the minimax condition

$$\max_{v\in Q}s'(t)f(t,x,u_e,v)=\min_{u\in P}\max_{v\in Q}s'(t)f(t,x,u,v)$$

$$(s(t)=\partial\varepsilon_0/\partial x)$$

According to Theorem 2. 1 of [3], if $\varepsilon_0(t_0,x_0)\leqslant 0$, then strategy $U_e^{(\vartheta)}$ solves the problem of the encounter of the motion of (1. 1) with set M at the instant $\vartheta=\vartheta^\circ$, where ϑ° is the instant of program absorption of set M by the motion of (1. 2) from the initial position $\{t_0,x_0\}$.

2. Example. Let the motion of the first pursuing object be described by the equations

$$y_1^{\cdot}=y_3,\quad y_2^{\cdot}=y_4 \tag{2. 1}$$

$$y_3^{\cdot}=\lambda y_3^2+u_1\cos\alpha-u_2\sin\alpha,\quad y_4^{\cdot}=u_1\sin\alpha+u_2\cos\alpha$$

$$y=\{y_1,\ldots y_4\},\quad u=\{0,0,u_1,u_2\}.$$

Here u is the control of the first pursuing player (object). The choice of the first player's control is subject to the constraint

$$u_1^2[t]+u_2^2[t]\leqslant\mu^2 \tag{2. 2}$$

There is a gap in the first player's control system; therefore, instead of the control $u[t]$ a certain control $u^*[t]$ acts on the system, where the vector $u^*[t]$ differs from the vector $u[t]$ by a rotation through some angle $\alpha[t]$; this obstacle must satisfy the constraint

$$|\alpha[t]|\leqslant\alpha_0<\pi/2 \tag{2. 3}$$

The motion of the second (pursued) player is described by the equations

$$z_1^{\cdot}=z_3,\quad z_2^{\cdot}=z_4,\quad z_3^{\cdot}=\lambda z_3^2+v_1,\quad z_4^{\cdot}=v_2 \tag{2. 4}$$

$$v=\{0,0,v_1,v_2\}$$

Here v is the control of the second player (object). The choice of the second player's control is subject to the constraint

$$v_1^2[t]+v_2^2[t]\leqslant\nu^2 \tag{2. 5}$$

We note that λ $(\lambda>0)$ is a small parameter in systems (2. 1) and (2. 4).

As the game's payoff we examine the quantity

$$\gamma[\vartheta]=[(y_1[\vartheta]-z_1[\vartheta])^2+(y_2[\vartheta]-z_2[\vartheta]^2]^{1/2} \tag{2. 6}$$

The first player's task is to minimize the quantity $\gamma[\vartheta]$; the second player's aim is to maximize the quantity $\gamma[\vartheta]$.

Let us now consider an auxiliary encounter game with payoff (2. 6), in which the players' behaviors are described by the equations in [5]. Consequently, the first player moves in accordance with Eqs. (2. 1) in which $\alpha=0$. Constraint (2. 2), in which the quantity μ

has been replaced by $\mu \cos \alpha_0$ is imposed on the choice of the first player's control. The second player's equations of motion and the constraint on his choice of controls coincide with (2.4) and (2.5), respectively.

Let us write the equations of the encounter game being investigated, as well as the equations for the auxiliary encounter game, as sixth-order systems by making a change of variables according to the formulas

$$x_1 = y_1 - z_1, \quad x_2 = y_2 - z_2, \quad x_3 = y_3, \quad x_4 = y_4, \quad x_5 = z_3, \quad x_6 = z_4$$

The choices of the controls in the encounter game under investigation are subject to constraints (2.2) and (2.5), respectively. The constraint on the gap imposed on the first player's control, is given by inequality (2.3). According to (2.6) the game payoff quantity $\gamma[\vartheta]$ takes the form $\gamma[\vartheta] \equiv 0$, i.e. the game termination set is given by the condition $M : \{x_1 = 0, x_2 = 0\}$.

The first player strives to insure that the point $\{x_1[t], x_2[t]\}$ hits onto the set $M : \{x_1 = 0, x_2 = 0\}$ at some instant $\vartheta = \vartheta^\circ$, while the second player tries to delay as long as possible the instant of point $\{x_1[t], x_2[t]\}$ hitting onto M. We note that in the encounter game being investigated the first player is opposed by the obstacle $\alpha[t]$ and by the second player choosing control $v[t]$, both being unknown to him. It is reasonable to assume that the second player chooses the function $\alpha[t]$ as well.

In the auxiliary system the first player's choice of control is subject to inequality (2.2) in which the quantity μ has been replaced by $\mu \cos \alpha_0$, while the second player's choice of control is subject to inequality (2.5). The game termination set M is the same as in the game being investigated. From [5] it follows that when $\lambda \leqslant \lambda_0$ (where λ_0 is some small number) the solution of the program maximin problem (1.5) for the auxiliary system is achieved by a unique pair of controls $\{u^\circ[t], v^\bullet[t]\}$. Here the set M is convex. Consequently, the point x_M of M, closest to point $x(\vartheta, t_0, x_0, u^\circ, v^\circ)$, is unique. Thus, the regular case of a game of encounter with set M holds for the auxiliary system. As is not difficult to be convinced, relation (1.6) is fulfilled for the auxiliary system and for the system being investigated. In such a case, taking into account the value of $\varepsilon_0(t, x, \vartheta, \lambda)$ computed in [5], we can construct, according to the theorem proved above, the first player's strategy $U_e^{(\vartheta)}$ guaranteeing that the motion of the system being investigated is led from the initial position $\{t_0, x_0\}$ onto set M at the instant ϑ_0. (The instant ϑ_0 is determined as the smallest positive root of the equation $\varepsilon_0(t_0, x_0, \vartheta, \lambda) = 0$.)

REFERENCES

1. Krasovskii, N. N., Extremal control in a nonlinear position differential game. Dokl. Akad. Nauk SSSR, Vol. 203, № 3, 1972.
2. Krasovskii, N. N. and Subbotin, A. I., Position Differential Games. Moscow, "Nauka", 1974.
3. Krasovskii, N. N., An encounter-evasion differential game, I, II. Izv. Akad. Nauk SSSR, Tekhn. Kibernetika, №№ 2, 3, 1973.
4. Pashkov, A. G., On one sufficient condition for games of encounter and evasion. Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, № 6, 1972.
5. Al'brekht, E. G., On the convergence of quasi-linear objects. PMM Vol. 34, № 4, 1970

Translated by N. H. C.